PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Connexion of Algebraic Functions with Automorphic Functions

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Phil. Trans. R. Soc. Lond. A 1899 **192**, 1-32
doi: 10.1098/rsta.1899.0001

PHILOSOPHICAL TRANSACTIONS.

I. *On the Connexion of Algebraic Functions with Automorphic Functions.*

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Received April 23,—Read May 12, 1898.

§ 1. *Introduction.*

It is well known that if

$$f(u, z) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the equation of an algebraic curve of genus (*genre*, *Geschlecht*) zero, then u and z can be expressed as rational functions of a single variable t. If, however, the genus of the curve (1) is unity, u and z can be expressed as uniform elliptic functions of a variable t.

The natural extension of these results was effected in 1881 by the discovery of automorphic functions; whatever be the genus of the curve (1), u and z can be expressed as uniform automorphic functions of a new variable.

This result is of great importance in the study of algebraic functions. Instead of taking z as the independent variable, and studying functions of z on the Riemann surface corresponding to the equation (1), we can take t as the independent variable, and consider the functions in the plane of t. We thus avoid the multiformity of the problem, and can apply the simpler and more developed theory of uniform functions.

Comparatively little of the published work on automorphic functions, however, has been written in connexion with the uniformisation of algebraic forms; in describing either groups applicable for the purpose, or the analytical connexions which exist between u, z, and t. The only automorphic functions known hitherto which have been applied to uniformise forms whose genus is greater than unity, are those given by certain sub-groups of the modular group (which will only uniformise special curves, containing no arbitrary constants), and those in which the fundamental polygon is the space outside a number of non-intersecting circles. These latter have been studied by SCHOTTKY,* WEBER,† and BURNSIDE,‡ and are capable of uniformising any algebraic form. As, however, the fundamental polygon is multiply-connected, the Abelian Integrals of the first kind, and the factorial functions associated with the algebraic form, are not uniform functions of the new variable.

With regard to the analytical connexion between the uniformising variable t and the variables u, z, of the algebraic form, POINCARÉ proved that if z is an automorphic function of t, then $\{t, z\}$ is another automorphic function of the same group, where $\{t, z\}$ is the Schwarzian derivative. t therefore satisfies a differential equation of the form

$$\{t, z\} = \phi(u, z),$$

where $\phi(u, z)$ is some rational function of u and z. SCHOTTKY and WEBER have determined $\phi(u, z)$, save for a number of undetermined constants, for the groups found by them, and KLEIN§ has obtained more general results, applying to any algebraic equation, but with a certain number of undetermined constants left in ϕ.

The problem has been formulated by KLEIN as one of conformal representation. The algebraic form which is given by

$$f(u, z) = 0$$

can be represented on a Riemann surface of class p, so that, corresponding to every pair of values (u, z) of the form, there is a place on the surface. By drawing $2p$ cuts we can make this surface simply-connected. Now let z be regarded as a function of a new variable t, having the following properties :—

1°. The dissected Riemann surface is to be conformally represented on a plane area in the t-plane, bounded by $4p$ curvilinear sides (namely, the conformal representations of the cuts, each cut giving two sides).

2°. Of the two sides of the t-area which correspond to any cut, one is to be derivable from the other by a projective substitution

$$\left(t, \frac{at + b}{ct + d}\right).$$

3°. The group formed by the combination and repetition of these $2p$ substitutions is to be discontinuous.

When a variable t has been found satisfying these conditions, u and z will be uniform automorphic functions of t; and we know by the existence-theorem of POINCARÉ and KLEIN that such a variable does exist, although the existence-theorem does not connect it analytically with z and u. The primary result of the present paper is, that the uniformisation of any algebraic form can be effected by automorphic func-

* 'Crelle,' vol. 101, 1897, p. 227.

† 'Göttinger Nachrichten,' 1886, p. 359.

‡ 'Proc. Lond. Math. Soc.,' vol. 23, 1891, p. 49.

§ 'Jahresbericht der Deutschen Mathematiker-Vereinigung,' 1894–5, p. 91.

tions of certain kinds of groups, which are described in § 3. These are either groups whose generating substitutions are of period two, or sub-groups of such groups. This theorem is made to depend on the well-known theorem that any algebraic form can, by birational transformation, be represented on a Riemann surface with only simple branch-points. A method is given for the division of the t-plane into polygons, corresponding to a group generated by real substitutions of period two, whose double points are not on the real axis; and the genus of the group is found. The group is of the kind called by POINCARÉ Fuchsian; the polygons into which the plane is divided are simply-connected, and cover completely the half of the t-plane which is above the real axis. Results are deduced relating to the possibility of uniformising any algebraic functions by automorphic functions of such groups, and the analytical connexion of the uniformising variable with the variables of the form.

In § 2, certain properties of substitutions of period two are found, which are of use later. These substitutions are for brevity termed "self-inverse" substitutions, owing to the fact that they are the same as their inverse substitutions.

In § 3, a method is given for carrying out the division of the plane into polygons, corresponding to a group generated by a given set of self-inverse substitutions. It is proved that the genus of the group is zero, although the group has sub-groups whose genus is greater than zero.

In § 4, the automorphic functions of the group are introduced. Since the group is of genus zero, these automorphic functions are all rational algebraic functions of one of them; the conformal representation of the polygons in the t-plane on the plane of this variable is considered. It is shown that the functions which have been obtained solve the following problem of conformal representation:—Draw from any point P, in the plane of a variable z, lines (not necessarily straight) to any other points A, B, C. . . . This set of rays is to be regarded as the boundary of the z-plane, and the problem is, to conformally represent the z-plane, thus bounded, on a simply-connected region in the plane of a variable t, in such a way that each of the lines PA, PB, PC, . . . gives rise to two distinct lines of the boundary of the t-region; and one of these lines is derivable from the other by a projective substitution

$$\left(t, \frac{at+b}{ct+d}\right).$$

The uniformisation of algebraic functions is afterwards made to depend on this problem of conformal representation.

In § 5, the analytical relation between the variables z and t is discussed. It is shown that they are connected by a differential equation which is a particular case of what has been named by KLEIN the "generalised LAMÉ's equation," and has been connected by BÔCHER with the differential equations of harmonic analysis.

In § 6, the functions which have been obtained are applied to the uniformising of algebraic forms. The differential equation in the hyperelliptic case is found to be the same as KLEIN'S "unverzweigt" differential equation for hyperelliptic forms, save that a number of constants left arbitrary in KLEIN'S equation are found to be zero. The conditions that $2p$ arbitrarily given substitutions may generate the group corresponding to a hyperelliptic equation of genus p are found.

In § 7, the consideration of the constants left undetermined in the differential equation of § 5 is resumed. If an algebraic form of genus p be given, the uniformising variable is one of ∞^{3p-3} variables, which are here termed "quasi-uniformising." Any quasi-uniformising variable affords a solution of the problem of conformally representing the Riemann surface of the form on a plane area whose sides are derived from each other in pairs by projective substitutions. The differential equations connecting the uniformising with the quasi-uniformising variables of a given algebraic form are obtained.

§ 2. *Properties of Self-inverse Substitutions.*

A projective substitution of a variable t is denoted by

$$\left(t, \frac{at+b}{ct+d}\right),$$

where we can always suppose that $ad - bc = 1$.

The substitutions, from which the groups considered in this paper are generated, are such that

$$a + d = 0.$$

Such a substitution is elliptic and of period two; its multiplier is -1, and it is its own inverse substitution. For brevity we shall call such substitutions "*self-inverse.*" Thus, if S denotes any self-inverse substitution, we have

$$S^2 = 1, \quad \text{and} \quad S = S^{-1}.$$

If T *be any substitution, and* S *be a self-inverse substitution, then* $T^{-1}ST$ *is a self-inverse substitution.* For the multiplier of a substitution is unaffected by the transformation which changes S into $T^{-1}ST$.

If there be any number of self-inverse substitutions, and a substitution be formed from them, then the substitution inverse to this is formed by taking the same substitutions in the reverse order. For if S_p, S_q, S_r, ..., are self-inverse substitutions, then obviously

$$S_pS_qS_r \ldots S_uS_vS_wS_wS_vS_u \ldots S_rS_qS_p = 1.$$

So if

$$T = S_pS_q \ldots S_uS_vS_w,$$

then

$$T^{-1} = S_wS_vS_u \ldots S_rS_qS_p.$$

The group formed by the combination and repetition of any two projective substitutions can be obtained as a self-conjugate sub-group of a group generated by three self-inverse substitutions.

For let

$$T_1 = \left(t, \frac{\alpha_1 t + \beta_1}{\gamma_1 t + \delta_1}\right) \quad \text{and} \quad T_2 = \left(t, \frac{\alpha_2 t + \beta_2}{\gamma_2 t + \delta_2}\right)$$

be the given substitutions; let

$$S_1 = \left(t, \frac{a_1 t + b_1}{c_1 t - a_1}\right), \quad S_2 = \left(t, \frac{a_2 t + b_2}{c_2 t - a_2}\right), \quad S_3 = \left(t, \frac{a_3 t + b_3}{c_3 t - a_3}\right)$$

be three self-inverse substitutions; then we have

$$S_3S_1(t) = \frac{(a_1a_3 + c_1b_3)\,t + (a_3b_1 - a_1b_3)}{(a_1c_3 - a_3c_1)\,t + (a_1a_3 + b_1c_3)}$$

and

$$S_3S_2(t) = \frac{(a_2a_3 + b_3c_2)\,t + (a_3b_2 - a_2b_3)}{(a_2c_3 - a_3c_2)\,t + (a_2a_3 + b_2c_3)}.$$

The equations to be satisfied by the coefficients of S_3, in order that we may have

$$S_3S_1 = T_1, \quad \text{and} \quad S_3S_2 = T_2,$$

reduce to

$$\left.\begin{aligned}(\alpha_1 - \delta_1)\,a_3 + \gamma_1 b_3 + \beta_1 c_3 = 0\\ (\alpha_2 - \delta_2)\,a_3 + \gamma_2 b_3 + \beta_2 c_3 = 0\end{aligned}\right\}.$$

These equations always admit of a solution for the ratios $a_3 : b_3 : c_3$, if the substitutions T_1 and T_2 are distinct. Thus, the substitution S_3 is determinate; and then S_1 and S_2 can be uniquely determined from the equations

$$S_1 = S_3T_1, \qquad S_2 = S_3T_2.$$

[Added June 2. In view of the subsequent limitations to substitutions for which $a^2 + bc$ is negative, it should be noticed that these equations may give either a positive or a negative value for $a^2 + bc$.]

Now let G denote the group formed from the generating substitutions S_1, S_2, S_3, and let H denote the group formed from the generating substitutions T_1 and T_2.

As T_1 and T_2 are themselves substitutions of the group G, the group H will be either the same as G, or a sub-group of it. We shall now show that H is a self-conjugate sub-group of G.

Since $S_r^2 = 1$, and $S_r = S_r^{-1}$, any substitution of G can be represented in the form

$$\Sigma = S_pS_qS_rS_s \ldots S_v, \quad \text{where} \quad p, q, r, s, \ldots v = 1, 2, 3.$$

But

$$S_1S_2 = T_1^{-1}T_2, \qquad S_1S_3 = T_1^{-1}, \qquad S_2S_1 = T_2^{-1}T_1,$$
$$S_2S_3 = T_2^{-1}, \qquad S_3S_1 = T_1, \qquad S_3S_2 = T_2.$$

Therefore every pair S_pS_q can be expressed in terms of T_1 and T_2.

So if the number of substitutions in Σ is even, the whole substitution can be expressed in the form

$$\Sigma = T_1^\alpha T_2^\beta T_1^\gamma T_2^\delta \ldots$$

i.e., it is a substitution of the group H.

But if the number of substitutions in Σ is odd, there will be one substitution S_v left at the end unpaired. Now

$$S_1 = T_1^{-1}S_3, \qquad S_2 = T_2^{-1}S_3, \qquad S_3 = S_3,$$

so in any case

$$\Sigma = T_1^\alpha T_2^\beta T_1^\gamma T_2^\delta \ldots T_r^\nu S_3.$$

So Σ is always either a substitution of H, or else the product of S_3 and a substitution of H.

Now let S_h be any substitution of H, and S_g any substitution of G.

Then $S_g^{-1}S_hS_g$ evidently contains, when decomposed into the substitutions S_1, S_2, S_3, an even number of them; for S_h contains an even number, and S_g^{-1} and S_g each contain the same number. Therefore $S_g^{-1}S_hS_g$ is a substitution of the group H; which establishes the required result, namely, that H is a self-conjugate sub-group.

As an example of this theorem, consider the modular group generated by the substitutions

$$(t,\ t+1) \quad \text{and} \quad \left(t,\ -\frac{1}{t}\right).$$

This is a self-conjugate sub-group of the group formed from the three self-inverse substitutions

$$(t,\ -t-1), \qquad \left(t, \frac{1}{t}\right), \qquad (t,\ -t).$$

As another example, take the group which occurs in the theory of elliptic functions, which is formed from the generating substitutions

$$(t,\ t+2w_1), \qquad (t,\ t+2w_2).$$

This is a self-conjugate sub-group of the group formed from the three self-inverse substitutions

$$(t,\ c-2w_1-t), \qquad (t,\ c-2w_2-t), \qquad (t,\ c-t)$$

where c is an arbitrary constant.

In this exceptional case, an arbitrary constant, c, is introduced. The reason is, that the quantities $\alpha_1 - \delta_1$, $\alpha_2 - \delta_2$, γ_1, γ_2, all vanish, so the two equations for determining $a_3 : b_3 : c_3$ reduce to the single equation

$$c_3 = 0.$$

Any group of substitutions which is formed from $(k+1)$ self-inverse substitutions as generating substitutions, always contains a self-conjugate sub-group which can be generated from k substitutions.

For let G be a group formed from $(k+1)$ self-inverse substitutions $S_1, S_2, S_3, \ldots S_{k+1}$. Then, as before, any substitution of G can be written in the form

$$\Sigma = S_p S_q S_r S_s S_t \ldots S_v.$$

Now let

$$T_1 = S_{k+1} S_1, \ T_2 = S_{k+1} S_2, \ \ldots \ T_k = S_{k+1} S_k.$$

Then

$$S_p S_q = S_p S_{k+1} S_{k+1} S_q = T_p^{-1} T_q.$$

Therefore, if the number of substitutions in Σ is even, Σ can be expressed in the form

$$\Sigma = T_p^{-1} T_q T_r^{-1} T_s \ldots T_v,$$

so Σ is a substitution of the group generated from $T_1, T_2, \ldots T_k$.

If the number of substitutions in Σ is odd, we have, therefore,

$$\Sigma = T_p^{\alpha} T_q^{\beta} \ldots T_s^{\delta} S_r,$$

and as

$$S_r = T_r^{-1} S_{k+1},$$

we have, in this case,

$$\Sigma = T_p^{\alpha} T_q^{\beta} \ldots T_s^{\delta} T_r^{-1} S_{k+1}.$$

So any substitution of the group G can be expressed either in the form Σ_p, or in the form $\Sigma_p S_{k+1}$, where Σ_p is a substitution of the group H, which is formed from $T_1, T_2, \ldots T_k$. And as in the case $k = 2$, which has been already discussed, we see that H is a self-conjugate sub-group of G.

[Added June 2, 1898.—H may, of course, coincide with G; I am indebted to Professor BURNSIDE for the example,

$$S_1^2 = 1, \ S_2^2 = 1, \ S_3^2 = 1, \ (S_1 S_2 S_3)^3 = 1,$$

in which this happens.]

To find the conditions that a group H, *generated from any k arbitrary projective substitutions,* $T_1, T_2, \ldots T_k$, *may in this way be a self-conjugate sub-group of a group* G *formed from $(k+1)$ self-inverse substitutions.*

Let

$$T_1 = \left(t, \frac{\alpha_1 t + \beta_1}{\gamma_1 t + \delta_1}\right), \quad T_2 = \left(t, \frac{\alpha_2 t + \beta_2}{\gamma_2 t + \delta_2}\right), \ldots T_k = \left(t, \frac{\alpha_k t + \beta_k}{\gamma_k t + \delta_k}\right).$$

Let

$$S_{k+1} = \left(t, \frac{at + b}{ct - a}\right), \quad \text{and let} \quad S_r = S_{k+1} T_r.$$

Then

$$S_r = \left(t, \frac{(a\alpha_r + b\gamma_r) t + (a\beta_r + b\,\delta_r)}{(c\alpha_r - a\gamma_r) t + (c\beta_r - a\,\delta_r)}\right).$$

If this is a self-inverse substitution, we have

$$a(\alpha_r - \delta_r) + b\gamma_r + c\beta_r = 0.$$

Thus the coefficients of the substitution S_{k+1} must satisfy the conditions

$$\left.\begin{array}{l} (\alpha_1 - \delta_1)\, a + \gamma_1 b + \beta_1 c = 0 \\ (\alpha_2 - \delta_2)\, a + \gamma_2 b + \beta_2 c = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (\alpha_k - \delta_k)\, a + \gamma_k b + \beta_k c = 0 \end{array}\right\}.$$

The elimination of $a : b : c$, from these equations gives $(k - 2)$ conditions between the coefficients of the substitutions T.

[Added June 2, 1898.—These conditions are sufficient, but are not actually necessary, as it may be possible to generate the group from a different set of substitutions, for which these conditions are satisfied, although they may not be satisfied by $T_1, T_2, \ldots T_k$.]

We shall, later, take $k = 2p$, and show that these $(2p - 2)$ conditions must be satisfied by the coefficients of $2p$ substitutions, whose group gives rise to automorphic functions which uniformise a hyperelliptic form of genus p.

§ 3. *The Division of the t-plane, corresponding to a group formed of Self-inverse Substitutions with Real Coefficients.*

A method will now be given for dividing the t-plane into regions, corresponding to a group generated from a given set of self-inverse substitutions. These regions are to be derivable from each other by applying the substitutions of the group.

Let

$$S = \left(t, \frac{at + b}{ct - a}\right)$$

be a self-inverse substitution with real coefficients a, b, c. Then the substitution transforms real values of t into other real values, so the real axis in the t-plane is unaffected by the substitution. If $(a^2 + bc)$ is negative, it is easily seen that the part of the t-plane above the real axis transforms into itself; if $(a^2 + bc)$ is positive, the part of the t-plane above the real axis transforms into the part below the real axis. We shall suppose that our groups are generated only from the former kind of substitutions, so we need only consider the half of the t-plane above the real axis.

Assuming then throughout that $(a^2 + bc)$ is negative for the substitution considered, it is obvious that the double points of the substitution are conjugate complex quantities; for the double points are the roots of the equation

$$ct^2 - 2at - b = 0.$$

Now draw any circle through the double points of the substitution. This circle cuts the real axis orthogonally.

Then *the substitution transforms the parts of the t-plane outside and inside this circle into each other.*

For, let the double points be

$$t = \gamma + i\delta, \qquad \text{and} \qquad t = \gamma - i\delta,$$

and let t' be the point into which any point t is transformed. Then the substitution may be written

$$\frac{t' - \gamma + i\delta}{t' - \gamma - i\delta} = - \frac{t - \gamma + i\delta}{t' - \gamma - i\delta}.$$

This shows that the angle subtended by t at the double points is changed into its supplement by the transformation; and therefore the circumferences of all circles through the double points transform into themselves, the part on one side of the double points transforming into the part on the other side of them. By considering the whole plane as made up of the circumferences of circles through the double points, we obtain the theorem.

Now consider the infinite group generated from a number $(n + 2)$ of these self-inverse substitutions,

$$S_1 = \left(t, \frac{a_1 t + b_1}{c_1 t - a_1}\right), \qquad S_2 = \left(t, \frac{a_2 t + b_2}{c_2 t - a_2}\right), \ldots S_{n+2} = \left(t, \frac{a_{n+2} t + b_{n+2}}{c_{n+2} t - a_{n+2}}\right),$$

which satisfy the relation

$$S_1 S_2 S_3 \ldots S_{n+2} = 1.$$

If $n = 1$, we find that it is impossible to satisfy this relation by self-inverse substitutions with conjugate complex double points; and if $n = 2$, it will be seen later that the method about to be given for the division of the plane into regions breaks down; but if $n > 2$, the relation can be satisfied, in an infinite number of ways, by substitutions of the required kind. A worked-out example is given below.

[Added June 2, 1898.—The possibility of the construction given below depends on the satisfying of certain inequalities among the constants of the substitutions; as in general, when the construction described is carried out, the sides of the polygon may cross each other.]

Now let $D_1, D_2, \ldots D_{n+2}$ be those double points, of the substitutions $S_1, S_2, S_3, \ldots S_{n+2}$ respectively, which are above the real axis.

Let C_1 be the point derived from D_{n+2} by applying the substitution S_1; or, as we can write it, let

$$C_1 = S_1 (D_{n+2}).$$

Similarly, let

$$C_2 = S_2 (C_1), \qquad C_3 = S_3 (C_2), \ldots, C_{n+1} = S_{n+1} (C_n).$$

Then

$$\begin{aligned} C_{n+1} &= S_{n+1} S_n \ldots S_2 S_1 (D_{n+2}) \\ &= S_{n+2} (D_{n+2}), \qquad \text{since } S_1 S_2 \ldots S_{n+2} = 1, \\ &= D_{n+2}. \end{aligned}$$

Now, by the last theorem, any point, and the point which is derived from it by a self-inverse substitution, lie on a circle through the double points of the substitution.

Therefore $D_{n+2} D_1 C_1$ lie on a circle orthogonal to the real axis.

Similarly $C_1 D_2 C_2$, $C_2 D_3 C_3, \ldots$, $C_n D_{n+1} C_{n+1}$, all lie on circles orthogonal to the real axis.

Therefore *a curvilinear polygon can be formed, whose* $(n + 1)$ *sides are arcs of circles orthogonal to the real axis and pass through the points* $D_1, D_2, D_3, \ldots D_{n+1}$, *respectively, and whose corners are the points* $D_{n+2}, C_1, C_2, \ldots C_n$.

Now suppose we transform the polygon by the substitution S_r, where $r = 1, 2, \ldots (n + 1)$. We obtain another polygon, likewise formed of arcs of circles orthogonal to the real axis, and having contact with the original polygon along the side $C_{r-1} D_r C_r$. The side of this new polygon which is the conformal representation of $C_{p-1} D_p C_p$ passes through the double points of the self-inverse substitution $S_r S_p S_r$; and on applying this substitution to the new polygon, we obtain a third polygon, having contact with the second along the side which is the conformal representation of $C_{p-1} D_p C_p$. In this way we can, as every new polygon is formed, surround it with other polygons, each having one side in common with it.

Now consider what happens at any angular point of the polygon, say D_{n+2}, when we derive polygons in this way. If we derive a fresh polygon by applying the substitution S_1, the derived polygon adjoins the original one along the side $D_{n+2} C_1$. If now we derive a fresh polygon from the original one by applying the substitution S_1S_2, this second derived polygon adjoins the first along its free side through D_{n+2}. If now again we derive a fresh polygon from the original one by applying the substitution $S_1S_2S_3$, this third derived polygon adjoins the second along its free side through D_{n+2}. Proceeding round D_{n+2} in this way, we obtain at last a polygon which is derived from the original one by the substitution $S_{n+1}S_n \ldots S_2S_1S_{n+1}S_n \ldots S_2S_1$.

But since

$$S_{n+1}S_n \ldots S_2S_1 = S_{n+2}, \quad \text{and} \quad S_{n+2}^2 = 1,$$

this is the identical substitution; in other words, *the* $2(n+1)^{\text{th}}$ *polygon as we go round* A *is the original polygon.*

In the same way we can prove, that at every corner $2(n+1)$ polygons meet. The sides of the polygons are all portions of circles orthogonal to the real axis. As we approach the real axis, the polygons become smaller and more crowded together.

If from the original polygon we derive others, by transforming it with all the substitutions of the group generated by $S_1, S_2, \ldots S_{n+2}$, *we cover the half-plane once and only once.* So the original polygon is a "fundamental region" for the group of substitutions.

In the annexed figure, the polygons in a portion of the plane are drawn to scale for the group formed from the substitutions

$$S_1 = \left(t,\ \frac{5t-74}{t-5}\right), \qquad S_2 = \left(t,\ \frac{2t-5}{t-2}\right), \qquad S_3 = \left(t,\ \frac{5t-29}{t-5}\right),$$
$$S_4 = \left(t,\ \frac{253t-2061}{33t-253}\right), \quad S_5 = \left(t,\ \frac{132t-1675}{11t-132}\right), \quad S_6 = \left(t,\ \frac{281t-4786}{17t-281}\right),$$

which are self-inverse substitutions satisfying the required relation

$$S_1S_2S_3S_4S_5S_6 = 1.$$

Here $n = 4$; the double points are given by

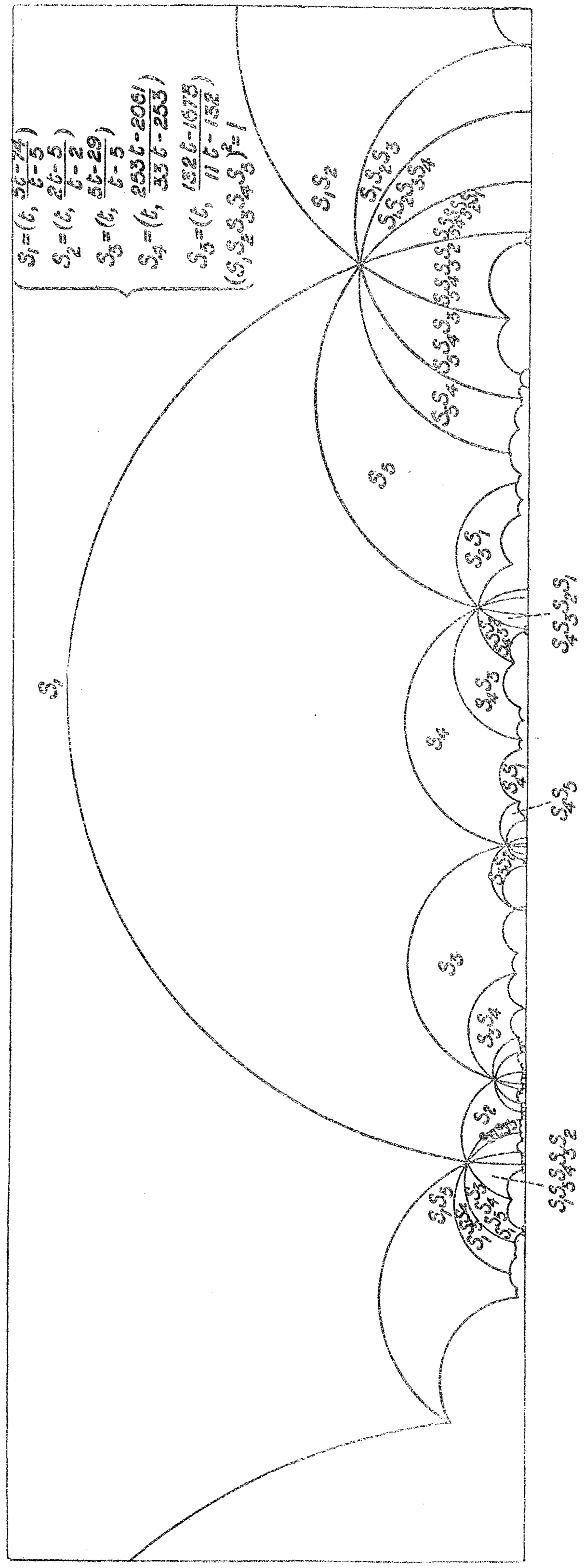
$$D_1 = 5 + 7i,\ D_2 = 2 + i,\ D_3 = 5 + 2i,\ D_4 = 7\tfrac{2}{3} + \sqrt{\tfrac{364}{99}}\,i,\ D_5 = 12 + \sqrt{\tfrac{91}{11}}\,i.$$

The vertex at the intersection of the S_1 and S_2 circles is at the point $t = 1 + i$.

Since the polygons are conformal representations of each other, they are equi-angular to each other.

From the construction of the polygon, all the angular points are equivalent in respect of the group.

The sum of the angles round any vertex is 2π; but these angles are the conformal representations of the angles of a polygon, taken twice. Hence *the sum of the angles of any polygon is* π.



Let $t = u + iv$; if we measure the distance between two points, in the non-Euclidian sense, by $\int \frac{|dt|}{v}$ taken along the circle orthogonal to the real axis and joining the points, then we can easily prove that the lengths of corresponding sides of the polygons are in this sense all equal; if we measure the area of any region by $\iint \frac{du\, dv}{v^2}$ taken over that region, we can show that the areas of all the polygons are also in this sense equal; and the areas and lengths of corresponding regions and lines in the polygons are all equal. *The substitutions by which the polygons are derived from each other are, in this non-Euclidian sense, simple displacements, which leave their dimensions unchanged.* All the theorems of LOBATCHEWSKI's geometry hold if, where LOBATCHEWSKI uses the word "straight line," we understand "circle orthogonal to the real axis."

Thus, in non-Euclidian phraseology, we can say that the network of polygons has been obtained by drawing a rectilinear polygon of $(n + 1)$ sides, deriving new polygons from it by turning the polygon through an angle π round the middle points of its sides, and deriving fresh polygons from these by the same process, until the whole non-Euclidian plane is covered. This enables us to see that *our figure is the natural extension of the division of a whole plane into parallelograms, so familiar in the theory of elliptic functions.* For that division can be obtained by drawing any rectilinear triangle in the Euclidian plane, deriving fresh triangles by turning it through an angle π round the middle points of its sides, and deriving new triangles from these by the same process, until the whole Euclidian plane is covered. The groups for which the elliptic functions are automorphic are sub-groups of the groups so obtained; and similarly the groups, whose automorphic functions are required in the uniformisation of algebraic forms of genus higher than unity, are sub-groups of the group we have found. The reason why we have to pass from Euclidian to non-Euclidian geometry is, that in the Euclidian plane it is impossible to obtain a rectilinear figure with more sides than three, the sum of whose angles is π.

If to the original polygon we apply the substitution S_{n+1}, the point D_{n+1} is unchanged, and the arcs $D_{n+2}D_{n+1}$ and $D_{n+1}C_n$ are transformed into each other. So the parts of the boundary of the polygon which correspond to each other in the transformations of the group are $D_{n+2}D_{n+1}$ to C_nD_{n+1}, C_nD_n to $C_{n-1}D_n$, . . ., C_1D_1 to $D_{n+2}D_1$, respectively. If now we suppose the polygon lifted up from the plane, and these corresponding arcs pieced together, we obtain a simple closed surface, without multiple connectivity.

Therefore *the genus* (genre, Geschlecht) *of the group* (as defined by POINCARÉ) *is zero.* The group however may have, and will in fact be proved to have, sub-groups whose genus is greater than zero.

§ 4. *The Automorphic Functions of the Group.*

From the fact which has just been proved, that the genus of all groups of the kind we have found is zero, we know that the algebraic relation between any two automorphic functions of the group is of genus zero; therefore *all the automorphic functions of the group can be expressed as rational functions of a certain one of them.* We shall denote this one by z.

First, let us see what degree of arbitrariness there is in the choice of the function z.

If a, b, c, d, are any four constants (which can without loss of generality be taken to satisfy the relation $ad - bc = 1$), then

$$\frac{az + b}{cz + d}$$

is another such function as z. Hence the function z contains three distinct arbitrary constants.

z takes every value once, and only once, in each polygon of the figure. The three arbitrary constants may be taken to be the place of its zero, the place of its infinity, and a multiplicative constant.

Now consider *the conformal representation of a t-polygon on the z-plane.*

The function z takes every value once in the polygon; therefore the conformal representation of the polygon will cover the whole z-plane. Also, z takes the same value, say e_{n+2}, at each of the corners of the polygon; suppose that z takes the values $e_1, e_2, e_3, \ldots e_{n+1}$, at the points $D_1, D_2, D_3, \ldots D_{n+1}$, respectively.

As t describes the boundary of the polygon, beginning at D_{n+2}, z begins with the value e_{n+2} and varies until, at D_1, the value e_1 is reached; then, retracing the same series of values, z returns to the value e_{n+2} at C_1. Then at D_2 the value e_2 is reached, and at C_2 z takes the value e_{n+2} again; and so on round the polygon.

Thus the conformal representation of the boundary of the polygon is a series of lines (not necessarily straight), radiating from the point e_{n+2} to the points $e_1, e_2, e_3, \ldots e_{n+1}$, in succession. *The polygon corresponds to the whole z-plane, with this regarded as boundary.* Small arbitrary variations in the form of the lines radiating from e_{n+2} to $e_1, e_2, \ldots e_{n+1}$, merely correspond to small arbitrary variations in the boundary of the polygon.

Thus *we see the nature of the solution of the problem: To conformally represent the whole plane of a variable z, bounded by a set of finite lines radiating from a point, on a curvilinear polygon in the plane of a variable t; this polygon being the fundamental region of an infinite discontinuous group of real projective substitutions of the variable t, and z being an automorphic function of the group.*

We may note that *dz/dt is zero at each of the double points.* For if t and t' are two points very near a double point, which are transformed into each other by the substitution corresponding to the point, we have approximately

$$dt' = - dt.$$

Thus dz/dt has values equal in magnitude, but opposite in sign, at the points t and t'; and therefore, making t and t' to coalesce in the double point, dz/dt is zero at the double point.

Let us now enumerate the constants at our disposal, in order to see the correspondence between the arrangement in the z-plane and the group of substitutions.

The t-figure is determined by $n+2$ self-inverse substitutions, $S_1, S_2, \ldots S_{n+2}$, satisfying the relation

$$S_1 S_2 S_3 \ldots S_{n+2} = 1. \quad \ldots \ldots \quad (1).$$

There are three real constants, a, b, c, in each substitution. But by reason of the relation

$$a^2 + bc = -1,$$

these three are only equivalent to two. Thus from the $(n+2)$ substitutions we get $(2n+4)$ real constants.

The relation (1) defines three of these constants in terms of the rest. Also, this group is not essentially different from one which is obtained by transforming it with any real substitution, which shows that three more of the constants are non-essential. So there are altogether $(2n-2)$ essential real constants involved in the t-figure.

Now considering the z-plane, there are $n+2$ points $e_1, e_2, \ldots e_{n+2}$; and each of these is defined by two real co-ordinates, giving $2n+4$ as the number of real constants. But we can make a homographic transformation of the plane, so as to transform any three of the points into three arbitrary points. This shows that 6 of the constants can be disregarded as non-essential. So we have $(2n-2)$ essential constants in the z-figure.

Hence *the number of essential constants is the same in the z-figure as in the t-figure.*

[ADDED June 2, 1898.—This does not in itself prove that for every z-figure there exists a corresponding t-figure; but the general existence-theorem of POINCARÉ and KLEIN can be applied to complete the proof.]

Hitherto we have derived the z-figure from the t-figure. The next section is chiefly concerned with the converse problem of deriving the t-figure from the z-figure.

§ 5. *The Analytical Relations between z and t.*

The analytical relations between z and t are of two kinds; (α) those which express z in terms of t and the constants of the substitutions, and (β) those which express t in terms of z and the quantities $e_1, e_2, \ldots e_{n+2}$.

The Thetafuchsian series of POINCARÉ solve the first problem for all classes of automorphic functions. We shall therefore only discuss relations of the kind (β).

As any quantity of the form $(at+b)/(ct+d)$, where a, b, c, d, are arbitrary real constants, is a solution of the problem (β) equally with t, we shall expect t to be given by a differential equation of which the general integral is $(at+b)/(ct+d)$; in other words, by a differential equation of the form

$$\tfrac{1}{2}\{t, z\} = \mathrm{R}(z),$$

where $\mathrm{R}(z)$ is some function of z, and

$$\{t, z\} = -\frac{dz^3/dt^3}{(dz/dt)^3} + \frac{3}{2}\frac{(dz^2/dt^2)^2}{(dz/dt)^4}$$

is a Schwarzian derivative.

As $\{t, z\}$ is unaltered by a change of t into $(at+b)/(ct+d)$, $\mathrm{R}(z)$ is an automorphic function of the group, and therefore $\mathrm{R}(z)$ *is a rational function of* z. We have to find $\mathrm{R}(z)$.

Considering the conformal representation, we see that z and t are regular functions of each other, except near the points $z = e_1, e_2, \ldots e_{n+2}, \infty$. Hence, except at these special points, $\frac{1}{2}\{t, z\}$ is a regular function of z, and we shall not get an infinity of $\mathrm{R}(z)$. As z is a uniform automorphic function of t, $\frac{dz}{dt}$ is infinite only at $z = \infty$.

Near $z = \infty$ (supposing for the present that no one of the quantities $e_1, e_2, \ldots e_{n+2}$, is infinite), z and t are uniform functions of each other, so

$$z = \frac{a}{t - t_0} + b + c(t - t_0) + \ldots, \text{ where } a \text{ is not zero.}$$

This gives

$$\tfrac{1}{2}\{t, z\} = \frac{3c}{a^3}(t - t_0)^4 + \ldots$$

Hence at $z = \infty$, $\frac{1}{2}\{t, z\}$ must be zero to at least the order $\frac{1}{z^4}$.

Near $z = e_r$, z is a uniform function of t, but dz/dt is zero. So near this point,

$$z - e_r = c(t - t_0)^2 + d(t - t_0)^3 + \ldots,$$

where c is not zero, since t has at the point a simple branch-point, considered as a function of z.

This gives

$$\tfrac{1}{2}\{t, z\} = \frac{3}{16c^2(t - t_0)^4} + \ldots \qquad = \frac{3}{16(z - e_r)^2} + \ldots$$

Thus the only infinities of the rational function $\mathrm{R}(z)$ are at the points $e_1, e_2, \ldots e_{n+2}$; and these points are poles of the kind just found.

Hence

$$\mathrm{R}(z) = \frac{3}{16}\sum_{r=1}^{n+2}\frac{1}{(z - e_r)^2} + \sum_{r=1}^{n+2}\frac{a_r}{z - e_r} + \mathrm{P}(z),$$

where P (z) is a polynomial in z, and a's are constants. Now at $z = \infty$ we must have a zero of at least the order $\frac{1}{z^4}$. Hence P $(z) = 0$; and since near $z = \infty$, R (z) can be expanded in the form

$$\mathrm{R}(z) = \tfrac{3}{16} \sum_{r=1}^{n+2} \left(\frac{1}{z^2} + \frac{2e_r}{z^3} + \ldots \right) + \sum_{r=1}^{n+2} \left(\frac{a_r}{z} + \frac{a_r e_r}{z^2} + \frac{a_r e_r^2}{z^3} + \ldots \right),$$

by equating to zero the coefficients of $\frac{1}{z}$, $\frac{1}{z^2}$, and $\frac{1}{z^3}$, respectively, we obtain

$$\sum_{r=1}^{n+2} a_r = 0,$$

$$\sum_{r=1}^{n+2} a_r e_r = -\frac{3(n+2)}{16},$$

$$\sum_{r=1}^{n+2} a_r e_r^2 = -\tfrac{3}{8} \sum_{r=1}^{n+2} e_r.$$

These conditions enable us to write R (z) in the form

$$\mathrm{R}(z) = \tfrac{3}{16} \sum_{r=1}^{n+2} \frac{1}{(z-e_r)^2} + \tfrac{3}{16} \frac{-(n+2)z^n + n\,.\,\Sigma e_r\,.\,z^{n-1} + c_1 z^{n-2} + \ldots + c_{n-1}}{(z-e_1)(z-e_2)\ldots(z-e_{n+2})},$$

where $c_1, c_2, \ldots c_{n-1}$ are constants as yet undetermined.

Hence *the required analytical relation between* t *and* z *is*

$$\tfrac{1}{2}\{t, z\} = \tfrac{3}{16} \sum_{r=1}^{n+2} \frac{1}{(z-e_r)^2} + \tfrac{3}{16} \frac{-(n+2)z^n + n\,.\,\Sigma e_r\,.\,z^{n-1} + c_1 z^{n-2} + \ldots + c_{n-1}}{(z-e_1)(z-e_2)\ldots(z-e_{n+2})}.$$

It will be seen that this is the differential equation for the quotient of two solutions of a linear differential equation of the second order with $(n + 2)$ singularities, at each of which the exponent-difference is $\frac{1}{2}$. Such linear differential equations have been studied by KLEIN,* as being the generalisation of LAMÉ's equation; and BÔCHER's book, 'Ueber die Reihenentwickelungen der Potential-Theorie' (Leipsic, TEUBNER, 1894), is chiefly concerned with them. BÔCHER proves that the differential equations of harmonic analysis are limiting cases of them.

We can *transform this equation to a simpler form.*

Put

$$w = \int \frac{dz}{\sqrt{(z-e_1)(z-e_2)\ldots(z-e_{n+2})}},$$

so w is a known function of z.

* 'Göttinger Nachrichten,' 1890, pp. 85–95.

Then the differential equation becomes

$$\tfrac{1}{2}\{t, w\} = \frac{n^2 - 4}{16} z^n - \frac{n(n-2)}{16} \sum_{r=1}^{n+2} e_r \,.\, z^{n-1} + d_1 z^{n-2} + d_2 z^{n-3} + \ldots + d_{n-1},$$

where $d_1, d_2, \ldots d_{n-1}$, are new undetermined constants replacing the c's.

This can be written

$$\tfrac{1}{2}\{t, w\} = \frac{n-2}{8(n+1)} \frac{d^3z/dw^3}{dz/dw} + k_1 z^{n-2} + k_2 z^{n-3} + \ldots + k_{n-1},$$

or

$$\tfrac{1}{2}\{t, w\} = \frac{n-2}{8(n+1)} \frac{1}{u} \frac{d^2u}{dw^2} + k_1 z^{n-2} + k_2 z^{n-3} + \ldots + k_{n-1} \quad . \quad . \quad . \quad (1),$$

where

$$u^2 = (z - e_1)(z - e_2) \ldots (z - e_{n+2}),$$

and where $k_1, k_2, \ldots k_{n-1}$ are new undetermined constants, replacing $d_1, d_2, \ldots d_{n-1}$.

If z has its infinity at a double point of one of the substitutions, we get a slightly different form of the equation.

In this case, one of the e's is infinite. Let $e_{n+2} = \infty$. Then, near $z = \infty$, the expansions are of the form

$$z = \frac{c}{(t - t_0)^2} + \ldots \quad \text{and} \quad \mathrm{R}(z) = \frac{3}{16z^2} + \ldots,$$

whence, by the same reasoning as before, we find that

$$\tfrac{1}{2}\{t, z\} = \tfrac{3}{16} \sum_{r=1}^{n+1} \frac{1}{(z - e_r)^2} + \tfrac{3}{16} \frac{-nz^{n-1} + c_1 z^{n-2} + \ldots + c_{n-1}}{(z - e_1)(z - e_2) \ldots (z - e_{n+1})}.$$

Put

$$w = \int \frac{dz}{\sqrt{(z - e_1)(z - e_2) \ldots (z - e_{n+1})}}.$$

Then the equation becomes

$$\tfrac{1}{2}\{t, w\} = \frac{n(n-2)}{16} z^{n-1} + d_1 z^{n-2} + d_2 z^{n-3} + \ldots + d_{n-1},$$

where again the quantities $d_1, d_2, \ldots d_{n-1}$, are undetermined constants.

This can be written

$$\tfrac{1}{2}\{t, w\} = \frac{n-2}{8(n+1)} \frac{d^3z/dw^3}{dz/dw} + k_1 z^{n-2} + k_2 z^{n-3} + \ldots + k_{n-1},$$

or,

$$\tfrac{1}{2}\{t, w\} = \frac{n-2}{8(n+1)} \frac{1}{u} \frac{d^2u}{dw^2} + k_1 z^{n-2} + k_2 z^{n-3} + \ldots + k_{n-1} \quad . \quad . \quad . \quad (2),$$

where

$$u^2 = (z - e_1)(z - e_2) \ldots (z - e_{n+1}).$$

The differential equations (1) *and* (2) *determine* t *in terms of* z *in the two cases respectively.*

The constants $k_1, k_2, \ldots k_{n-1}$, are as yet undetermined. The reason is, that we have not yet made any use of the condition which in fact does determine them; namely, that all the projective substitutions, which t undergoes when the independent variable z of the differential equation describes a circuit round one of the singularities, are such as to leave unchanged a certain circle. This circle is, in the figure we have drawn, the real axis of the variable t, which is unchanged by all the substitutions of the group; but it may more generally be any circle in the t-plane. This condition will be shown in § 7 to be equivalent to the determination of $(n-1)$ complex quantities, which are the constants $k_1, k_2, \ldots k_{n-1}$. But a further consideration of this is deferred to § 7. For the present we shall suppose $k_1, k_2, \ldots k_{n-1}$ determined in such a way as to give the required representation.

§ 6. *Application of the Preceding Theory to the Uniformising of Algebraic Forms.*

We have proved that the genus of groups of the kind we have found is zero, and hence the automorphic functions of the group as it stands will not uniformise algebraic forms whose genus is greater than zero. But we can find sub-groups of the original group, and these will be found to be of genus greater than zero.

The process of deriving these sub-groups is analogous to the method of building up a Riemann surface of any genus by superposing a number of plane sheets and connecting them along branch lines. We join together a certain number of the polygons in the figure, and regard them as forming one new polygon. This will, in certain cases, be the fundamental polygon of a sub-group of the original group, and may have a genus greater than zero.

Consider a double polygon, made up by taking together the original polygon, and the polygon derived from it by transforming with the substitution S_{n+1}, and erasing the boundary which separates them. The new polygon has $2n$-sides. By erasing all the lines corresponding to the line already erased, we obtain a division of the half-plane into $2n$-gons. The opposite sides of the $2n$-gon are easily seen to be transformed into each other by the n substitutions

$$T_1 = S_{n+1}S_1,\ T_2 = S_{n+1}S_2, \ldots T_n = S_{n+1}S_n,$$

respectively.

This $2n$*-gon is a "fundamental region" for the group generated from the substitutions* $T_1, T_2, \ldots T_n$. We proved in § 2 that the group generated by $T_1, T_2, \ldots T_n$, is a self-conjugate sub-group of the group formed by $S_1, S_2, \ldots S_{n+2}$; and that any substitution of the latter group is equivalent to a substitution of the former group acting on either the identical substitution or on S_{n+1}. This corresponds to the fact that a point in any of the derived $2n$-gons can be obtained by transformation with the substitutions T from a point in either the original $(n + 1)$-gon or the $(n + 1)$-gon derived from this by the substitution S_{n+1}.

We have, therefore, obtained a new division of the half-plane into $2n$-gons, and found the group of substitutions corresponding to it. We can now find the genus p of this group.

The opposite sides of the $2n$-gon are transformed into each other by substitutions of the group. If we suppose the $2n$-gon lifted up from the plane, and opposite sides pieced together, we obtain a surface of connectivity $(n + 1)$. If n is even, this surface is of genus p where $n = 2p$. In what follows we shall suppose n even.

Hence, *the algebraic relation between any two automorphic functions of this group is, in general, of genus* $p = \frac{1}{2}n$.

The function z, which has been obtained, takes every value once in each $(n + 1)$-gon; and therefore it takes every value twice in each $2n$-gon. But this is the condition that the algebraic form, made up of the automorphic functions of the group, should be hyperelliptic.

Hence, *the algebraic form, which is made up of the automorphic functions of the group, is hyperelliptic, and of genus* $\frac{1}{2}n$; and, as z is a variable which takes every value twice in each polygon, the form consists of rational functions of z and u, where u is a function of z defined by an equation

$$u^2 = (z - a_1)(z - a_2) \ldots (z - a_{n+2}),$$

where $a_1, a_2, \ldots a_{n+2}$ are constants to be determined. But the function

$$\sqrt{(z - e_1)(z - e_2) \ldots (z - e_{n+2})}$$

is an automorphic function of the group, for it has the same value, save for a change of sign, at corresponding points in adjacent $(n + 1)$-gons, and therefore the same value at corresponding points in different $2n$-gons.

Hence

$$a_1 = e_1, \ a_2 = e_2, \ldots a_{n+2} = e_{n+2},$$

and we see that the automorphic functions of the group generated from the substitutions $T_1, T_2, \ldots T_n$ are the algebraic functions of the form defined by the equation

$$u^2 = (z - e_1)(z - e_2) \ldots (z - e_{n+2}).$$

Thus we have the solution of the problem, "*To find a variable of which the functions rational on the Riemann surface of the equation*

$$u^2 = (z - e_1)(z - e_2) \ldots (z - e_{n+2})$$

are uniform functions."

We could have foreseen this by regarding the problem as one of conformal representation. The algebraic functions can be regarded as uniform functions on a Riemann surface which covers the z-plane twice, the branch-points being at the points $e_1, e_2, \dots e_{n+2}$. Now join the point e_{n+2} to each of the points $e_1, e_2, e_3, \dots e_{n+1}$. Then each of the sheets, regarded as an infinite plane bounded by these lines, is represented conformally on one of the $(n+1)$-gons in the t-plane; by taking two adjacent $(n+1)$-gons, we obtain a $2n$-gon, which corresponds to the fact that by taking the two z-planes, and connecting them along the line $e_{n+2}\,e_{n+1}$, we obtain the Riemann surface as dissected by n cross-cuts.

The analytical connexion between the variables in a hyperelleptic form and the uniformising variable t is therefore given by the equations of § 5. It can be shown that the differential equation found there is, as might be expected, one of KLEIN's* "unverzweigt" differential equations for hyperelliptic forms. It can be obtained by equating $(p-2)$ of the arbitrary constants in KLEIN's equation to zero.

There are p integrals of the first kind connected with the form. It is easily proved that if v is one of them, then v undergoes a projective substitution of the form

$$(v,\ c - v),$$

where c is a constant, when t is transformed by one of the generating substitutions of the group.

The theory of Abelian integrals of the form can be developed with t as independent variable; but developments of this kind are outside the scope of this paper.

One consequence of the results just obtained is that we can find *the conditions that* $2p$ *arbitrarily given projective substitutions may generate the group corresponding to a hyperelliptic equation of genus* p.

Let the substitutions be $T_1, T_2 \dots T_{2p}$, where

$$T_r = \left(t,\ \frac{a_r t + b_r}{c_r t + d_r}\right).$$

On comparing the results of this section with those of § 2, we see that *the conditions may be expressed in the form*

$$\begin{vmatrix} a_r - d_r & b_r & c_r \\ a_s - d_s & b_s & c_s \\ a_t - d_t & b_t & c_t \end{vmatrix} = 0, \quad (r, s, t, = 1, 2, 3, \dots 2p).$$

[Added June 3, 1898.—These conditions are not, however, proved to be strictly necessary, since the group may be generated by another set of substitutions to which these conditions apply, although they do not apply to $T_1, T_2, \dots T_{2p}$. And the

* 'Göttinger Nachrichten,' 1890, p. 85.

inequalities expressing the conditions that the sides of the generating polygon do not cross must also be satisfied.]

In all our work hitherto it has been assumed that $p > 1$. *The case $p = 1$ is exceptional;* algebraic forms of genus unity cannot be uniformised by groups of the kind we have found. For if the construction which has been given were possible for $p = 1$, we should have, as the fundamental polygon of the group, a triangle whose sides are, in the non-Euclidian sense, straight lines, and the sum of whose angles is π. But this is impossible, for in LOBATCHEWSKI's geometry the sum of the angles of a triangle is always less than π. When the sum is equal to π we arrive at the limiting case of Euclidian geometry. Therefore the construction fails, and we have to devise instead a construction in which Euclidian geometry replaces non-Euclidian. We take four substitutions, S_1, S_2, S_3, S_4, satisfying the relation

$$S_1S_2S_3S_4 = 1,$$

which are self-inverse and leave the Euclidian absolute unchanged, *i.e.*, which are all of the type

$$(t, \quad c - t),$$

where c is a complex constant. By reasoning exactly analogous to that in § 3, we see that these substitutions generate a group, to which corresponds a division of the plane into rectilinear triangles. The sub-group which is got by taking adjacent triangles in pairs gives a division of the plane into parallelograms; and this is the well-known group of the doubly-periodic functions, which uniformise algebraic curves of genus unity.

The following shows how the former construction breaks down in this case.

If possible, let S_1, S_2, S_3, S_4, be four self-inverse substitutions with real coefficients satisfying the relation

$$S_1S_2S_3S_4 = 1.$$

Then if

$$S_r = \left(t, \ \frac{a_r t + b_r}{c_r t - a_r}\right),$$

we have

$$S_1S_2S_3\,(t) = \frac{(a_1a_2a_3 + a_1b_2c_3 + a_3b_1c_2 - a_2b_1c_3)\,t + (a_1a_2b_3 - a_1b_2a_3 - b_1c_2b_3 + b_1a_2a_3)}{(c_1a_2a_3 + c_1b_2c_3 - a_1c_2a_3 + a_1a_2c_3)\,t + (c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - a_1a_2a_3)}.$$

This has to be a self-inverse substitution, since S_4 is self-inverse.

So

$$a_1b_2c_3 + b_1a_3c_2 - a_2b_1c_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 = 0,$$

or

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Let $\gamma_r + i\delta_r$ and $\gamma_r - i\delta_r$ be the double points of S_r, then

$$a_r = \gamma_r, \quad b_r = -(\gamma_r^2 + \delta_r^2), \quad c_r = 1.$$

Therefore

$$\begin{vmatrix} \gamma_1^2 + \delta_1^2 & \gamma_2^2 + \delta_2^2 & \gamma_3^2 + \delta_3^2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

This shows that the double points of all three substitutions lie on a circle orthogonal to the real axis. Since $S_2S_3S_4$ is a self-inverse substitution, the double points of S_4 lie on the same circle.

Hence, if we attempt to construct the fundamental polygon, we find that all its angular points lie on the same circle orthogonal to the real axis, and therefore all its sides coalesce, and its area is zero. This explains why the method fails in this case.

We now proceed to *the uniformisation of algebraic forms which are not hyper-elliptic.* These only occur when the genus is greater than two.

If we are given any algebraic form of genus p, it is known that it can by birational transformation be represented on a Riemann surface of which all the branch-points are simple, *i.e.*, only two sheets interchange at any branch-point.

Let $f(u, z) = 0$ be an algebraic equation corresponding to this surface. Suppose the branch-points are at the values of z for which $z = e_1, e_2, e_3, \ldots e_{n+2}$, respectively. It may of course happen that for some of these values of z there are several branch-points superposed on each other on the Riemann surface.

Now in the z-plane, join the point e_{n+2} to each of the points $e_1, e_2, \ldots e_{n+1}$, and conformally represent this, in the plane of a variable t, on the fundamental polygon of a group from $(n + 2)$ self-inverse substitutions, as before explained.

Then, as before, z is a uniform function of t. At each of the points $z = e_1, e_2, \ldots e_{n+2}$, say e_r, u is expansible in a series of ascending powers of either $(z - e_r)^{\frac{1}{2}}$ or $(z - e_r)$, according as the point $z = e_r$ happens to be a branch-point or not in the sheet in which the point is situated. But near this point $(z - e_r)^{\frac{1}{2}}$ is expansible in a power-series in terms of $(t - t_0)$, where t_0 is the value of t at the point; so in either case, u is expansible as a series of ascending powers of $(t - t_0)$; that is, u has no branch-point, considered as a function of t, at this point.

But since z is a uniform function of t, the only points where u can have branch-points, considered as a function of t, are the points where u has branch-points considered as a function of z; that is, the points $e_1, e_2, \ldots e_{n+2}$. Hence, u is a uniform function of t.

Thus, *any algebraic curve can be uniformised by means of groups of substitutions formed from self-inverse substitutions.*

It will be seen that a great similarity exists between the place occupied by self-inverse substitutions, in the theory of groups of projective substitutions, and the place occupied by branch-points at which only two branches interchange, in the theory of Riemann surfaces; the usefulness of the method of self-inverse substitutions depends on the fact that algebraic forms can be represented on Riemann surfaces with only simple branch-points.

Algebraic functions are not, however, the only ones which can be uniformised. POINCARÉ* has proved a general existence-theorem that, if $u_1, u_2, \ldots u_m$, are any multiform analytical functions of a variable z, a variable t always exists, such that $z, u_1, u_2, \ldots u_m$, are uniform functions of t. The existence-theorem, however, does not connect t analytically with the other variables. If $u_1, u_2, \ldots u_m$, are transcendental functions of z, their multiformity will not in general be capable of being expressed by simple branch-points, and so the groups generated by self-inverse substitutions cannot be used.

§ 7. *The Undetermined Constants in the Differential Equation connecting z and t.*

In § 5, certain constants $k_1, k_2, \ldots k_{n-1}$, in the differential equation connecting z and t, were left undetermined. It was there explained that they are to be determined by the consideration that the group of substitutions of t leaves unchanged a fundamental circle. In general, however, arbitrary constants occurring in similar differential equations cannot be determined by this consideration, as the group may be "Kleinian," *i.e.*, it may not conserve a fundamental circle. The following discussion approaches the subject from this more general point of view.

The Riemann surface, corresponding to the algebraic form $f(u, z) = 0$, can be made simply-connected by drawing $2p$ cuts, and the problem of finding the uniformising variable t can be divided into two parts, as follows :—

1. Finding all the variables τ, which are such that the dissected Riemann surface is represented on the τ-plane by a curvilinear polygon, whose $4p$ sides can be derived from each other in pairs by projective substitutions of τ.

2. Selecting from among these variables τ, a variable t, which is such that the group generated from these projective substitutions is a discontinuous group.

We shall call the variables τ *quasi-uniformising* variables, to distinguish them from the true uniformising variable t.

In the case of the groups we have found, the differential equation of § 5 *gives the quasi-uniformising variables; the determination of* $k_1, k_2, \ldots k_{n-1}$ *is equivalent to selecting the uniformising variable from among them.*

In this section the connexion between the uniformising and quasi-uniformising variables is considered for more general groups.

As an example of the nature of quasi-uniformising variables, take the algebraic equation

$$u^2 = 4z^3 - g_2 z - g_3.$$

* 'Bulletin de la Société Math. de France,' 1883, vol. 11, p. 112.

To this corresponds a Riemann surface of two sheets, which can be resolved by two cuts into a simply-connected surface.

Let P be the Weierstarssian elliptic function associated with this curve; and w_1, w_2, its periods.

Consider u and z as functions of τ, where

$$u = \mathrm{P}'(\log \tau), \qquad z = \mathrm{P}(\log \tau).$$

In the τ-plane, form a curvilinear parallelogram ABCD, of which the side CB is derived from AD by the projective substitution

$$(\tau, \ e^{w_1}\tau),$$

and the side CD is derived from AB by the projective substitution

$$(\tau, \ e^{w_2}\tau).$$

Then within this parallelogram ABCD, the dissected Riemann surface corresponding to the curve

$$u^2 = 4z^3 - g_2 z - g_3$$

is conformally represented; the sides AD, CB of the parallelogram correspond to the two edges of one cross-cut, and the sides AB, CD to the other; and, as we have seen, the opposite sides of the parallelogram are derived from each other by projective substitutions. But in spite of this, u and z are not uniform functions of τ. The reason is, that τ is only a quasi-uniformising variable; when we derive all possible polygons from ABCD by applying the group of substitutions generated from

$$(\tau, \ e^{w_1}\tau) \quad \text{and} \quad (\tau, \ e^{w_2}\tau),$$

the polygons so derived cover the plane more than once.

The connexion between the uniformising and quasi-uniformising variables for any algebraic form is given by the following theorems.

If t is a uniformising variable of an equation

$$f(u, z) = 0,$$

and T *is any holomorphic Thetafuchsian function of t of order two, then the quotient of any two solutions of the differential equation*

$$\frac{d^2v}{dt^2} + \mathrm{T}v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is a quasi-uniformising variable.

The term "holomorphic Thetafuchsian function of order two" may require some explanation.

Let $\left(t, \dfrac{at+b}{ct+d}\right)$ be any one of the substitutions of the group associated with the given uniformising variable t. Then a Thetafuchsian function T of order m is such that

$$T\left(\frac{at+b}{ct+d}\right) = (ct+d)^{2m} T(t).$$

We have said that T is to be holomorphic (except at the singularities of the group). Such functions exist; for instance, if w be an Abelian integral of the first kind associated with the curve, then dw/dt is a holomorphic Thetafuchsian function of order one, and $(dw/dt)^2$ is a holomorphic Thetafuchsian function of order two.

To prove the theorem, let

$$\tau = v_1/v_2,$$

where v_1 and v_2 are any two solutions of (1). Then v_1 and v_2 have singularities, considered as functions of t, only where T has singularities. But in any one of the polygons in the t-plane, T has no singularities. Therefore, v_1 and v_2 are holomorphic functions of t (except at the essential singularities of the group, which for the present we do not consider).

Also, v_1 and dv_1/dt cannot be zero together at any point; for if they were, by equation (1), v_1 would be permanently zero. Similarly for v_2.

Therefore, at all points p within any one of the polygons in the t-plane, we have expansions beginning with

$$v_1 = c + d(t - t_0) + \dots,$$

where c and d are not both zero, and

$$v_2 - e + f(t - t_0) + \dots,$$

where e and f are not both zero.

And we may not have d and f zero together, as v_1 and v_2 are independent solutions of the differential equation.

So, at all points except the singularities of the group,

$$\tau = \frac{c + d(t - t_0) + \dots}{e + f(t - t_0) + \dots}$$

gives either

$$\tau = A + B(t - t_0) + \dots,$$

or,

$$\tau = A(t - t_0) + B(t - t_0)^2 + \dots,$$

or,

$$\tau = \frac{A}{t - t_0} + B + C(t - t_0) + \dots$$

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In all these cases t and τ are uniform functions of each other, near the point considered. So u and z are, near the point, uniform functions of τ. This is easily seen to be true also of $t = \infty$.

Now, let accented letters denote the effect of operating on t with a substitution

$$\left(t, \frac{at+b}{ct+d}\right)$$

of the group.

We have

$$\frac{d^2v'}{dt'^2} + T'v' = 0.$$

Now $T' = (ct+d)^4 T$. Write $v' = \frac{\xi}{ct+d}$.

Then

$$\frac{d^2v'}{dt'^2} = (ct+d)^2 \frac{d}{dt}\left\{(ct+d)^2 \frac{d}{dt}\left(\frac{\xi}{ct+d}\right)\right\} = (ct+d)^3 \frac{d^2\xi}{dt^2}.$$

Therefore

$$(ct+d)^3 \frac{d^2\xi}{dt^2} + (ct+d)^3 T\xi = 0,$$

or

$$\frac{d^2\xi}{dt^2} + T\xi = 0.$$

So $\xi = Av_1 + Bv_2$, where A and B are constants, and

$$v' = \frac{Av_1 + Bv_2}{ct+d}.$$

Therefore

$$\tau' = \frac{v_1'}{v_2'} = \frac{A_1v_1 + B_1v_2}{A_2v_1 + B_2v_2},$$

or

$$\tau' = \frac{A_1\tau + B_1}{A_2\tau + B_2}.$$

This shows that, when t is transformed by a projective substitution of the group, τ is transformed by a corresponding projective substitution

$$\left(\tau, \frac{A_1\tau + B_1}{A_2\tau + B_2}\right).$$

Thus the theorem is proved, namely, that the dissected Riemann surface can be conformally represented on a polygon in the τ-plane, and the sides of this polygon can be derived from each other in pairs by certain projective substitutions; in other words, τ is a quasi-uniformising variable. An infinite number of variables τ can be got in this way, for T depends linearly on several arbitrary constants.

In the above theorem, for the sake of simplicity, we have made a restriction which is really unnecessary, namely, we have supposed that t is a uniformising variable. *t can, however, be any quasi-uniformising variable* if we make the corresponding extension in the meaning of T. T will now have to be a function of t, which is holomorphic in any of the polygons, and which obeys the law

$$T\left(\frac{at+b}{ct+d}\right) = (ct+d)^4\, T(t)$$

for substitutions of the group generated from the substitutions which change the sides of the t-polygon into each other. Such functions exist; for, as before, if w is an Abelian integral of the first kind connected with the curve, $(dw/dt)^2$ is such a function. T is, of course, really a multiform function of t, if t is a quasi-uniformising variable; but as it is not possible to pass from one of its values to another by any paths contained within one of the polygons, we can regard it as uniform within that polygon. The proof in this extended case is just as before. Thus we have the more general theorem:

If t is any uniformising or quasi-uniformising variable of an algebraic form

$$f(u, z) = 0,$$

and T *is any holomorphic Thetafuchsian function of t of order two, then the quotient of any two solutions of the differential equation*

$$\frac{d^2v}{dt^2} + Tv = 0$$

is another uniformising or quasi-uniformising variable.

To complete the theorem, we must prove that the converse is also true. Suppose, then, that τ and t both belong to the set of uniformising and quasi-uniformising variables, so that a polygon in the τ-plane corresponds to a polygon in the t-plane, point for point, and to each of the substitutions of the group $\left(\tau, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$ corresponds a substitution $\left(t, \frac{at+b}{ct+d}\right)$.

Now τ is the quotient of two integrals of the equation

$$\frac{d^2v}{dt^2} + Tv = 0$$

if

$$T = -\frac{1}{2}\frac{d^3t/d\tau^3}{(dt/d\tau)^3} + \frac{3}{4}\frac{(d^2t/d\tau^2)^2}{(dt/d\tau)^4}.$$

Now τ has no branch-point, considered as a function of t, and t has no branch-point, considered as a function of τ, except at the limiting points of the groups. So, if we consider any point in the t-plane, which is not one of the singularities of the group, $dt/d\tau$ and $d\tau/dt$ are, in its vicinity, regular functions of t.

So T is holomorphic at all points except the singularities of the group.

Now, denoting as before the effect of a substitution of the group by accents, we have

$$\begin{aligned} T' &= -\tfrac{1}{2}\frac{d^3t'/d\tau'^3}{(dt'/d\tau')^3} + \tfrac{3}{4}\frac{(d^2t'/d\tau'^2)^2}{(dt'/d\tau')^4} \\ &= \left(\frac{dt}{dt'}\right)^2\left[-\tfrac{1}{2}\frac{d^3t/d\tau^3}{(dt/d\tau)^3} + \tfrac{3}{4}\frac{(d^2t/d\tau^2)^2}{(dt/d\tau)^4}\right] \\ &= (\gamma t + \delta)^4\, T. \end{aligned}$$

So, T is a function of t of the kind already specified.

So, *the converse of the theorem is true.*

Thus, *if we can find any one quasi-uniformising variable of an algebraic form, we can find the totality of all uniformising and quasi-uniformising variables by this equation.*

We can now *find the functions* T.

If

$$t' = \frac{at + b}{ct + d},$$

we have

$$\frac{dt'}{dt} = \frac{1}{(ct + d)^2},$$

and so

$$(dz/dt')^2 = (ct + d)^4\,(dz/dt)^2.$$

Thus $(dz/dt)^2$ is a Thetafuchsian function of order two; any other Thetafuchsian function of order two can be written in the form

$$T = R(z, u)\,.\,(dz/dt)^2,$$

where $R(z, u)$ is an automorphic function of the group, *i.e.*, a rational function of the algebraic form.

If the algebraic form is of genus p, it is known* that any function $R(z, u)$ for which T is holomorphic is a linear function of $(3p - 3)$ special functions. These we can write

$$R_1(z, u),\quad R_2(z, u), \ldots R_{3p-3}(z, u).$$

The case $p = 1$ is exceptional; here there is one such function, T, namely, a constant.

* HUMBERT, 'LIOUVILLE'S Journal,' (4), vol. 2, p. 239, 1886.

In general, therefore, we have

$$T = [a_1 R_1 (z, u) + a_2 R_2 (z, u) + \ldots + a_{3p-3} R_{3p-3} (z, u)] (dz/dt)^2,$$

where $R_1 (z, u)$, $R_2 (z, u), \ldots R_3 (z, u)$ are functions which can be found, and $a_1, a_2, \ldots a_{3p-3}$, are arbitrary constants.

We can now *find the form of the differential equation which gives all the quasi-uniformising variables.* Take any quasi-uniformising variable τ of the algebraic equation

$$f(u, z) = 0.$$

For it, we have

$$\tfrac{1}{2} \{\tau, z\} = \phi (z, u),$$

where ϕ is some rational function of z and u.

If t is the most general quasi-uniformising variable, we have seen that t is given as the quotient of two solutions of the differential equation

$$d^2v/dt^2 + Tv = 0,$$

where

$$T = [a_1 R_1 (z, u) + a_2 R_2 (z, u) + \ldots + a_{3p-3} R_{3p-3} (z, u)] (dz/dt)^2.$$

Hence

$$\tfrac{1}{2} \{t, \tau\} = T.$$

But

$$\{t, z\} = \{\tau, z\} + (d\tau/dz)^2 \{t, \tau\}.$$

Therefore

$$\tfrac{1}{2} \{t, z\} = \phi (z, u) + T (d\tau/dz)^2,$$

or

$$\tfrac{1}{2} \{t, z\} = \phi (z, u) + a_1 R_1 (z, u) + a_2 R_2 (z, u) + \ldots + a_{3p-3} R_{3p-3} (z, u).$$

Thus, *the solution of the problem of finding all the variables t, which will conformally represent the Riemann surface of a given algebraic form on a curvilinear polygon, whose sides are derived from each other in pairs by projective substitutions, is given by a differential equation containing $(3p - 3)$ arbitrary parameters linearly, and the problem of finding the uniformising variable is equivalent to that of determining these parameters in order that that group generated by these substitutions may be discontinuous.*

Now let us return to the differential equation of § 5, which we can write

$$\tfrac{1}{2} \{\tau, w\} = \frac{n-2}{8(n+1)} \frac{1}{u} \frac{d^2u}{dw^2} + k_1 z^{n-2} + k_2 z^{n-3} + \ldots + k_{n-1}.$$

If we take any set of values $k_1, k_2 \ldots k_{n-1}$ for the undetermined constants, this differential equation will give a variable τ in terms of z, which will not in general be the variable t of §§ 3 and 4. But the variables τ so found will solve the problem of conformally representing the z-plane, regarded as bounded by a number of finite lines radiating from a point, on a curvilinear polygon in the τ-plane, such that the sides of the boundary can be transformed into each other in pairs by certain projective substitutions. The variable t is one of these variables, characterised by the condition that the infinite group generated from these substitutions is a discontinuous group.

We can, in fact, find the functions T in this case. We must have

$$\mathrm{T} = \mathrm{R}\,(z)\left(\frac{dz}{dt}\right)^2,$$

and R (z) must be such that T is holomorphic. So the only possible poles of R (z) are the places where dz/dt is zero, *i.e.*, the places $z = e_1, e_2, \ldots e_{n+2}$. At these places dz/dt is zero of the first order : so $(dz/dt)^2$ is zero of the second order, and R (z) may have a pole of the second order.

Therefore

$$\mathrm{R}\,(z) = \frac{\mathrm{I}\,(z)}{u^2},$$

where

$$u^2 = (z - e_1)(z - e_2)\ldots(z - e_{n+2}),$$

and I (z) is an integral function of z. At $z = \infty$, dz/dt has a pole of the second order, and u^2 a pole of the $(n+2)^{\text{th}}$ order. So I (z) may have a pole of the $(n-2)^{\text{th}}$ order.

Therefore

$$\mathrm{I}\,(z) = k'_1 z^{n-2} + k'_2 z^{n-3} + \ldots + k'_{n-1}$$

and

$$\mathrm{T} = \frac{k'_1 z^{n-2} + k'_2 z^{n-3} + \ldots + k'_{n-1}}{u^2}\left(\frac{dz}{dt}\right)^2.$$

Thus if τ is the quotient of two solutions of the equation

$$d^2v/dt^2 + \mathrm{T}v = 0$$

and t is defined by the equation

$$\tfrac{1}{2}\,\{t, z\} = \mathrm{R}\,(z),$$

then τ is defined by the equation

$$\tfrac{1}{2}\,\{\tau, z\} = \mathrm{R}\,(z) + \frac{k'_1 z^{n-2} + k'_2 z^{n-3} + \ldots k'_{n-1}}{u^2}.$$

Comparing this with the equation of § 5, we see that *the variables t given by it, when the constants $k_1, k_2, \ldots k_{n-1}$, are arbitrary, are the quasi-uniformising variables.*

We can now prove that the number of conditions which have to be satisfied in order that the group of substitutions of t may be discontinuous, *i.e.*, in this case may conserve a fundamental circle, is equal to the number of the constants k.

In order that a self-inverse substitution with complex coefficients,

$$\left(t, \frac{at + b}{ct - a}\right),$$

may leave unchanged a given circle, two of the four real constants contained in the substitution must be determinate in terms of the others.

Now there are $(n + 2)$ fundamental self-inverse complex substitutions, containing $4(n + 2)$ real constants; of these, the relation

$$S_1 S_2 S_3 \ldots S_{n+2} = 1$$

accounts for six. So $(2n + 1)$ of the real constants are determined in terms of the other $(2n + 1)$ by the condition that the group is to conserve a fundamental circle; but as the fundamental circle may be any whatever, and so involves three constants, we must deduct three from the number of equations, giving $(2n - 2)$. Thus, $2n - 2$ real, or $n - 1$ complex, constants can be determined from the condition that the substitutions of t conserve a fundamental circle. *This accords with the fact, otherwise arrived at, that the constants $k_1, k_2, \ldots k_{n-1}$, in the differential equation have to be determined from this consideration.*

Among the quasi-uniformising variables of any algebraic form there are several distinct uniformising variables. The groups we have found in § 3 have simply-connected fundamental polygons. But automorphic functions exist, for which the fundamental polygons are multiply-connected.

The simplest example of such a function is

$$z = P\left(\frac{iw_1}{\pi} \log t\right),$$

where P is Weierstrass' elliptic function with periods $2w_1$ and $2w_2$; the fundamental polygon is the space between two circles in the t-plane.

The automorphic functions studied by Schottky, Weber, and Burnside may be regarded as generalisations of this. As these uniformising variables with multiply-connected fundamental polygons are included in the general set of quasi-uniformising variables, they are defined by the same differential equations as the uniformising variables with simply-connected polygons, except that the constants k will have different values.